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POISSON COMPLEXES AND SUBELLIPTICITY

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0. Introduction

Let

$$E^0 \xrightarrow{D} E^1 \xrightarrow{D} \cdots \xrightarrow{D} E^N \tag{0.1}$$

be a complex of first-order differential operators on a (compact) n-dimensional manifold X, and let

$$0 \to E^0 \xrightarrow{\sigma(D)(x,\zeta)} E^1 \xrightarrow{\sigma(D)(x,\zeta)} \cdots \xrightarrow{\sigma(D)(x,\zeta)} E^N \to 0 \tag{0.2}$$

be the associated top-symbol complex on $T^*X/\{0\}$. If the bundles E^i are given Hermitian metrics we may define the adjoint operators D^*.

Definition 0.1. We say that (0.1) is $\frac{1}{2}$-subelliptic at position i if and only if an estimate

$$\|u\|_{1/2} \leq C_K\{\|D_i u\|_0 + \|D^*_{i-1}u\|_0 + \|u\|_0\}\,, \quad u \in C_0^\infty(K, E^i) \tag{0.3}$$

holds for each compact subset $K \subseteq X$. Here the norms are Sobolev norms.

Hörmander [7] showed that the estimate (0.3) depends only on the behavior of the top-symbol complex (0.2) in the neighborhood of the characteristic variety, and, in fact, is equivalent to certain "test-estimates" at each point of the characteristic variety.

Guillemin [4], [5], by introducing the notion of asymptotic derivative, was able to reformulate the test-estimates of Hörmander as L^2-exactness of certain asymptotic test-complexes associated to (0.2). Using this new formulation of $\frac{1}{2}$-subellipticity he was able to show that $\frac{1}{2}$-subellipticity is independent of the choice of Hermitian metrics for the bundles E^i (despite the fact that the adjoint operators D^* are defined in terms of the Hermitian metrics), and that $\frac{1}{2}$-subellipticity is invariant under formal pseudo-differential conjugations of the origi-

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nal complex (0.1). He then showed that if (0.1) is formally exact in the sense of Goldschmidt [3], i.e., is a Spencer complex, then, in the neighborhood of a Cohen-Macaulay point of the characteristic variety, (0.1) can be almost-conjugated into a direct sum of a symbol-exact complex and a so-called Poincaré complex. (Theorem 2 in [5] asserting the existence of a full formal conjugation is probably a little too strong.) The symbol-exact complex is automatically $\frac{1}{2}$-subelliptic and the Poincaré complex, in the case of simple characteristics, is $\frac{1}{2}$-subelliptic if and only if a Levi-form criterion holds.

In this paper we show that the study of $\frac{1}{2}$-subellipticity for a Spencer complex is carried out most naturally by restricting attention to the top-symbol complex and by working always over the cotangent space rather than over the base. We also show that this is the natural setting for the normal-form decomposition of the Spencer complex.

In §1 we introduce the notion of Poisson complex on a symplectic manifold M and show how to associate the notion of $\frac{1}{2}$-subellipticity to a Poisson complex. The top-symbol complex (0.2) is Poisson, M being $T^*X/\{0\}$, and the notions of $\frac{1}{2}$-subellipticity for (0.1) and for the Poisson complex (0.2) coincide in this case. We show that if a Poisson complex is a direct sum of two subcomplexes, then each of these subcomplexes is Poisson, and, moreover, that the original Poisson complex is $\frac{1}{2}$-subelliptic if and only if each of the subcomplexes is $\frac{1}{2}$-subelliptic.

In §2 we give a new proof, differing in essential points from that of Guillemin, that the top-symbol complex of a Spencer complex can, in a conic neighborhood in $T^*X/\{0\}$ of a Cohen-Macaulay point, be written as a direct sum of two subcomplexes, one exact and the other a top-symbol Poincaré complex. Note that the bundles occurring in these subcomplexes are not required to be pull-backs to $T^*X/\{0\}$ of bundles on X. By the results of §1, to analyze $\frac{1}{2}$-subellipticity for the Spencer complex it suffices to analyze $\frac{1}{2}$-subellipticity for an exact Poisson complex and for a top-symbol Poincaré Poisson complex. Thus we are able to conclude, in §3, with the Levi-form criterion for $\frac{1}{2}$-subellipticity in the case of simple characteristics. We remark that in this top-symbol, cotangent space approach there is no need to perform formal pseudo-differential conjugations. Indeed, invariance under formal pseudo-differential conjugation and independence of choice of Hermitian metrics for our bundles over X is essentially equivalent to independence of choice of Hermitian metrics over $T^*X/\{0\}$ for our bundles.

We are grateful to Professor Victor Guillemin for making available to us unpublished manuscripts of his work and for several helpful conversations. In particular, the proof of Lemma 1.10 came out of a conversation with him.

Various details which are omitted here may be found in the unpublished notes of Guillemin [4] or in the author's thesis [9].

We conclude this section by noting that W. J. Sweeney [10] has recently given a new formulation of r-subellipticity, in particular $\frac{1}{2}$-subellipticity, which, among other things, shows independence of choice of Hermitian metrics and invariance under formal pseudo-differential conjugation.

1. Poisson complexes and $\frac{1}{2}$-subellipticity

Let

$$E^0 \xrightarrow{p^0} E^1 \xrightarrow{p^1} \cdots \xrightarrow{p^N} E^N \tag{1.1}$$

be a *complex* of bundle maps on a symplectic manifold M, i.e.,

$$p^{i+1} \cdot p^i = 0 \,. \tag{1.2}$$

Choose local canonical coordinates $(x_1, \cdots, x_n, \xi_1, \cdots, \xi_n)$ and local frames for our bundles.

Definition 1.1. The complex (1.1) is a *Poisson complex* if and only if there exist bundle maps $q^i: E^i \to E^{i+1}$ such that

$$\sum_j \frac{\partial p^{i+1}}{\partial \xi_j} \frac{\partial p^i}{\partial x_j} = q^{i+1}p^i + p^{i+1}q^i \,. \tag{1.3}$$

Lemma 1.2. *The condition that* (1.1) *be a Poisson complex is intrinsic, i.e., does not depend on the choice of local canonical coordinates or local frames.*

Proof. To show independence of choice of frames we replace p^{i+1}, p^i by $\bar{p}^{i+1} = rp^{i+1}s$, $\bar{p}^i = s^{-1}p^it$, where r, s, t are invertible matrices. By a direct expansion of $\sum_j(\partial\bar{p}^{i+1}/\partial\xi_j)(\partial\bar{p}^i/\partial x_j)$ and using the identities

$$\frac{\partial p}{\partial \xi_j} p + p \frac{\partial p}{\partial \xi_j} = 0 \,, \tag{1.4}$$

$$\frac{\partial p}{\partial x_j} p + p \frac{\partial p}{\partial x_j} = 0 \,, \tag{1.5}$$

which follow from differentiating (1.2), we see that if (1.3) holds for some q^{i+1}, q^i, then there exist $\bar{q}^{i+1}, \bar{q}^i$ such that

$$\sum_j \frac{\partial \bar{p}^{i+1}}{\partial \xi_j} \frac{\partial \bar{p}^i}{\partial x_j} = \bar{q}^{i+1}\bar{p}^i + \bar{p}^{i+1}\bar{q}^i \,. \tag{1.6}$$

To show invariance under change of local canonical coordinates it suffices to show that condition (1.3) is equivalent to the condition:

There exist $\bar{q}^{i+1}$, $\bar{q}^i$ such that

$$\{p^{i+1}, p^i\} = \bar{q}^{i+1}p^i + p^{i+1}\bar{q}^i \,, \tag{1.7}$$

where the left side denotes the Poisson bracket of $p^{i+1}, p^i = \sum_j(\partial p^{i+1}/\partial\xi_j) \cdot(\partial p^i/\partial x_j) - (\partial p^{i+1}/\partial x_j)(\partial p^i/\partial\xi_j)$; for Poisson brackets are invariant under change of local canonical coordinates. To show (1.3) is equivalent to (1.7) we use the identity

$$\frac{\partial^2 p}{\partial\xi_j\partial x_j} p + \frac{\partial p}{\partial\xi_j}\frac{\partial p}{\partial x_j} + \frac{\partial p}{\partial x_j}\frac{\partial p}{\partial\xi_j} + p\frac{\partial^2 p}{\partial\xi_j\partial x_j} = 0\,, \tag{1.8}$$

which we get by twice differentiating (1.2). Multiplying (1.3) by 2 and subtracting (1.8), we can take $\bar{q} = 2q + \partial^2 p/\partial\xi_j\partial x_j$. In the same way (1.3) follows from (1.8) by taking $q = \frac{1}{2}(\bar{q} - \partial^2 p/\partial\xi_j\partial x_j)$.

Remark. If $p, \sqrt{-1}\, q$ are the first two terms in the asymptotic expansion of a pseudo-differential operator P (see Example 1 below), then $\bar{q}$ defined above is roughly the subprincipal part of P. See [8] for the definition.

Example 1. If

$$E^0 \xrightarrow{P} E^1 \xrightarrow{P} \cdots \xrightarrow{P} E^N \tag{1.9}$$

is a complex of (classical) pseudo-differential operators on a manifold X, then the top-symbol complex

$$E^0 \xrightarrow{p} E^1 \xrightarrow{p} \cdots \xrightarrow{p} E^N \tag{1.10}$$

on $T^*X/\{0\}$ is Poisson.

Proof. Let $P \sim p + q + \cdots$ be the asymptotic expansion associated to P for some choice of local coordinates and local frames. Since $P^2 = 0$, all the terms in the asymptotic expansion of P^2 must equal 0. But

$$P^2 \sim p^2 + \left(pq + qp - \sqrt{-1}\sum \frac{\partial p}{\partial\xi_j}\frac{\partial p}{\partial x_j}\right) + \cdots\,. \tag{1.11}$$

Thus $p^2 = 0$ and $pq + qp - \sqrt{-1}\sum(\partial p/\partial\xi_j)(\partial p/\partial x_j) = 0$. q.e.d.

In particular, (0.2) is a Poisson complex.

Example 2. Let $\{p_\alpha\}_{\alpha=1,\cdots,q}$ be a set of scalar functions on M, and $\{p, E^i\}$ be the associated top-symbol Poincaré complex (see Definition 2.1). Here E^0 is taken to be the trivial line bundle over M.

Lemma 1.3. *The scalar top-symbol Poincaré complex $\{p, E^i\}$ is a Poisson complex if and only if $\{p_\alpha, p_\beta\}$ is in the ideal generated by the p_α's.*

Proof. ($\Rightarrow$) is obvious.

($\Leftarrow$). Assume $\{p_\alpha, p_\beta\} = \sum_\gamma p_\gamma s_{\alpha\beta\gamma}$. Then one verifies by direct computation that $\sum(\partial p/\partial\xi_j)(\partial p/\partial x_j) : \bigwedge^i \to \bigwedge^{i+2}$ is given by

$$\underline{w} \mapsto \left(\sum_{\substack{\alpha<\beta \\ \gamma}} p_\gamma s_{\alpha\beta\gamma} w_\alpha \wedge w_\beta\right) \wedge \underline{w}\,. \tag{1.12}$$

Define $s\colon \bigwedge^1 \to \bigwedge^2$ by

$$s\colon w_\gamma \mapsto \sum_{\alpha<\beta} s_{\alpha\beta\gamma} w_\alpha \wedge w_\beta \;.$$

Then, setting $p' = \sum p_\alpha w_\alpha \in \bigwedge^1$, we see that $s(p') = \sum_{\substack{\alpha<\beta \\ \gamma}} p_\gamma s_{\alpha\beta\gamma} w_\alpha \wedge w_\beta$, and so (1.12) is given by

$$w \mapsto s(p') \wedge w \;. \tag{1.12$'$}$$

Define $q^{i+1}\colon \bigwedge^{i+1} \to \bigwedge^{i+2}$ by

$$\begin{aligned} &q^{i+1}(w_{\alpha(1)} \wedge \cdots \wedge w_{\alpha(i+1)}) \\ &\quad = \sum_{j=1}^{i+1} (-1)^{j-1} w_{\alpha(1)} \wedge \cdots \wedge s(w_{\alpha(j)}) \wedge \cdots \wedge w_{\alpha(i+1)} \;. \end{aligned} \tag{1.13}$$

It is easy to see that this is antisymmetric in $w_{\alpha(1)}, \cdots, w_{\alpha(i+1)}$, and therefore is well-defined.

Moreover, it follows from the definition (1.13) that

$$q^{i+1}(p' \wedge \underline{w}) + p' \wedge q^i(\underline{w}) = s(p') \wedge \underline{w} \tag{1.14}$$

for $\underline{w} \in \bigwedge^i$. That is,

$$q^{i+1}p + pq^i = \sum \frac{\partial p}{\partial \xi_j} \frac{\partial p}{\partial x_j} \;. \tag{1.14$'$}$$

q.e.d.

We need to show next that the notion of $\frac{1}{2}$-subellipticity can be intrinsically defined for a Poisson complex; but first we need the notion of intrinsic derivative, due to Porteous [1], and its generalization, the asymptotic derivative of Guillemin.

Let E, F be vector bundles over a manifold M (not necessarily symplectic), $A\colon E \to F$ a bundle map, and x a point in M.

Lemma 1.4. *For a tangent vector X to M at x there is a well-defined map*

$$A_X\colon \text{Kernel}\, A_x \to F_x/\text{Image}\, A_x \;. \tag{1.15}$$

We call the map (1.15) the *intrinsic derivative* of A with respect to X.

Proof. Choose local frames for E and F. Then it is meaningful to differentiate the resulting matrix A with respect to X, and we thus get a map $A'\colon E_x \to F_x$. If we restrict A' to Kernel A_x and follow by the projection of F_x onto $F_x/$Image A_x, it is easy to see that the resulting map A_X is independent of our choice of local frames. q.e.d.

Now let $\lambda = \{(x_i, \lambda_i)\}$ be a sequence in $M \times \boldsymbol{R}^+$ such that $x_i \to x$ and $\lambda_i \to \infty$. We associate to the bundle map $A\colon E \to F$ and the sequence λ the following subspaces K_λ and R_λ of E_x and F_x, respectively:

(1.16)
$$K_\lambda = \{e \in E_x \mid \exists\, e_i \in E_{x_i} \text{ and } f \in F_x \text{ such that } e_i \to e \text{ and } \lambda_i A_{x_i} e_i \to f\}\,,$$

(1.17)
$$R_\lambda = \{f \in F_x \mid \exists\, e_i \in E_{x_i},\, e_i/\lambda_i \to 0 \text{ and } A_{x_i} e_i \to f\}\,.$$

Definition 1.5. We say that λ is an *asymptotic sequence* if $K_\lambda = K_{\lambda'}$ and $R_\lambda = R_{\lambda'}$ for every subsequence λ' of λ. If λ is an asymptotic sequence, then K_λ is called the *asymptotic kernel* and R_λ the *asymptotic image*.

One shows easily that every sequence μ has an asymptotic subsequence λ.

Lemma 1.6. *For an (asymptotic) sequence λ with respect to the bundle map $A: E \to F$ there is a well-defined map*

(1.18)
$$A_\lambda\colon K_\lambda \to F_x/R_\lambda\,.$$

We call the map (1.18) the *(asymptotic) derivative* of A with respect to λ.

Proof. For e in K_λ choose $e_i \to e$ such that $\lambda_i A_{x_i} e_i \to f$, some element of F_x. We then set $A_\lambda e = [f]$, the class of f in F_x/R_λ. The map is well-defined since the difference vanishes modulo R_λ if we choose a different sequence $e'_i \to e_i$ with $\lambda_i A_{x_i} e'_i \to f'$. q.e.d.

Consider now a complex of bundle maps

(1.19)
$$\cdots \to E \xrightarrow{A} F \xrightarrow{A} G \to \cdots\,.$$

We say that λ is asymptotic for the complex if it is asymptotic for each map A of the complex.

It is easy to prove the following lemma.

Lemma 1.7 *The asymptotic derivative A_λ induces a map (which we continue to call A_λ) on the asymptotic homology groups $H_\lambda = K_\lambda/R_\lambda$:*

(1.20)
$$\cdots \to H_\lambda(E) \xrightarrow{A_\lambda} H_\lambda(F) \xrightarrow{A_\lambda} H_\lambda(G) \to \cdots\,.$$

Moreover the sequence (1.20) is a complex, i.e.,

(1.21)
$$A_\lambda^2 = 0\,.$$

We also have

Lemma 1.8. *The intrinsic derivative A_X induces a map (which we continue to call A_X) on the asymptotic homology groups H_λ:*

(1.22)
$$\cdots \to H_\lambda(E) \xrightarrow{A_X} H_\lambda(F) \xrightarrow{A_X} H_\lambda(G) \to \cdots\,.$$

Moreover, the sequence (1.22) is a complex, i.e.,

(1.23)
$$A_X^2 = 0\,.$$

It is also easy to show that the following relations hold (where X and Y are both tangent vectors at x):

$$A_X A_Y + A_Y A_X = 0\ , \tag{1.24}$$

$$A_\lambda A_X + A_X A_\lambda = 0\ . \tag{1.25}$$

We return now to the Poisson complex (1.1). Let $p_\lambda\colon H_\lambda \to H_\lambda$ be the asymptotic derivative (1.20), and, choosing local canonical coordinates $(x_1, \cdots, x_n, \xi_1, \cdots, \xi_n)$, let

$$B_{\lambda,i}\colon H_\lambda \to H_\lambda\ , \tag{1.26}$$

$$C_{\lambda,i}\colon H_\lambda \to H_\lambda \tag{1.27}$$

be the maps $p_{\partial/\partial\xi_i}, p_{\partial/\partial x_i}$, respectively, of (1.22), where $\partial/\partial\xi_i$ and $\partial/\partial x_i$ denote the tangent vectors associated to the coordinate functions ξ_i and x_i.

Consider now the differential operators

$$\mathscr{P}_\lambda = p_\lambda + \sum_{i=1}^{n} B_{\lambda,i}\frac{1}{\sqrt{-1}}\frac{\partial}{\partial y_i} + C_{\lambda,i}y_i \tag{1.28}$$

from $C_0^\infty(\boldsymbol{R}^n, H_\lambda)$ to $C_0^\infty(\boldsymbol{R}^n, H_\lambda)$. Here $(y_1, \cdots, y_n)$ are coordinates for $\boldsymbol{R}^n$. Note that $p_\lambda, B_{\lambda,i}$, and $C_{\lambda,i}$ are constants as far as y-dependence is concerned. We next prove a lemma which will be of critical importance in showing that $\frac{1}{2}$-subellipticity for (1.1) is independent of the choice of Hermitian metrics for the bundles E^i.

Lemma 1.9. *The assumption that* (1.1) *is a Poisson complex implies* (*and, in fact, is essentially equivalent to*) *the following*:

The sequence of differential operators

$$C_0^\infty(\boldsymbol{R}^n, H_\lambda(E^0)) \xrightarrow{\mathscr{P}_\lambda} C_0^\infty(\boldsymbol{R}^n, H_\lambda(E^1)) \xrightarrow{\mathscr{P}_\lambda} \cdots \xrightarrow{\mathscr{P}_\lambda} C_0^\infty(\boldsymbol{R}^n, H_\lambda(E^N)) \tag{1.29}$$

is a complex, i.e., $\mathscr{P}_\lambda\cdot\mathscr{P}_\lambda = 0$.

Proof. A direct computation shows that the identity $\mathscr{P}_\lambda\cdot\mathscr{P}_\lambda = 0$ is equivalent to the identities (1.21), (1.23), $\cdots$, (1.25), which follow simply from the fact that (1.1) is a complex, together with the identity $\sum_i B_{\lambda,i}C_{\lambda,i} = 0$, which follows immediately from (1.3), i.e., the fact that (1.1) is Poisson. q.e.d.

We shall next show

Lemma 1.10. *Up to unitary equivalence, the complex* (1.29) *does not depend on the choice of local canonical coordinates* $(x_1, \cdots, x_n, \xi_1, \cdots, \xi_n)$.

Assuming this for the moment, we may conclude

Proposition 1. *If* $\{p, E^i\}$ *is a Poisson complex on a symplectic manifold* M, *then given any asymptotic sequence* λ *one can associate to* $\{p, E^i\}$, *in a completely intrinsic fashion, a complex of first-order differential operators* $\{\mathscr{P}_\lambda, C_0^\infty(\boldsymbol{R}^n, H_\lambda(E^i))\}$.

Proof of Lemma 1.10. We shall only sketch the main points of the argument, since the details become tedious. One sees easily that it suffices to prove the following:

Given a scalar function r on M and (x, ξ) and (x', ξ') two sets of local canonical coordinates at $m \in M$, there exists a unitary operator U such that the diagram

$$\begin{array}{ccc} \mathscr{S} & \xrightarrow{R_{(x,\xi)}} & \mathscr{S} \\ U\downarrow & & \downarrow U \\ \mathscr{S} & \xrightarrow{R_{(x',\xi')}} & \mathscr{S} \end{array} \tag{1.30}$$

commutes. Here $\mathscr{S} \subseteq L^2(\boldsymbol{R}^n, \boldsymbol{C})$ is the Schwartz space of rapidly decreasing functions, and the differential operator $R_{(x,\xi)}$ is defined by

$$R_{(x,\xi)} = \sum_i \frac{\partial r}{\partial \xi_i}\bigg|_m \frac{1}{\sqrt{-1}} \frac{\partial}{\partial y_i} + \frac{\partial r}{\partial x_i}\bigg|_m y_i \,. \tag{1.31}$$

Since the coefficients of $R_{(x,\xi)}$ depend only on the first derivatives of r at a single point m, it suffices to consider the case of $(x, \xi) \to (x', \xi')$ a *linear* canonical transformation. Aside from a simple permutation of the indices in $x_1, \cdots, x_n$ (and the same permutation of $\xi_1, \cdots, \xi_n$), the simplest type of canonical transformation is of the form

$$x'_i = \xi_i\,, \quad \xi'_i = -x_i\,, \quad \text{when} \quad i \in I\,;$$
$$x'_i = x_i\,, \quad \xi'_i = \xi_i\,, \quad \text{when} \quad i \notin I\,,$$

where I is some subset of $\{1, \cdots, n\}$. We call such transformations elementary canonical transformations. Since $\widehat{(1/\sqrt{-1})(\partial/\partial y_i)f}(\eta) = \eta_i \hat{f}(\eta)$ and $\widehat{y_i f}(\eta) = -(1/\sqrt{-1})(\partial/\partial \eta_i)\hat{f}(\eta)$ where $\wedge$ denotes the Fourier transform in the i-th variable, we see that, in the case of an elementary canonical transformation, we can make (1.30) commute by taking U to the Fourier transform in the variables I. U is unitary by the Plancherel theorem.

Caratheodory [2] shows that every canonical transformation can be written as the product of an elementary canonical transformation and a canonical transformation $(x, \xi) \to (x', \xi')$ given by a generating function $Q(x, \xi')$. The generating function satisfies the conditions

$$x' = \partial Q/\partial \xi'\,, \qquad \xi = \partial Q/\partial x\,, \tag{1.32 a}$$

$$\partial^2 Q/\partial x \partial \xi' \text{ is nonsingular}\,. \tag{1.32 b}$$

Hence to conclude our proof of the lemma it suffices to find a unitary U making (1.30) commute in case $(x, \xi) \to (x', \xi')$ is given by (1.32).

We define $\tilde{U} : \mathscr{S} \to \mathscr{S}$ by

$$(\tilde{U}f)(z) = K \int e^{-iQ(y,z)} f(y) dy \; . \tag{1.33}$$

Here K is a constant to be determined later. We shall show that

$$\begin{array}{ccc} \mathscr{S} & \xrightarrow{R_{(x,\xi)}} & \mathscr{S} \\ {\scriptstyle \tilde{U}} \downarrow & & \downarrow {\scriptstyle \tilde{U}} \\ \mathscr{S} & \xrightarrow{R_{(\xi',-x')}} & \mathscr{S} \end{array} \tag{1.34}$$

commutes. Thus by taking U equal to $\tilde{U}$ followed by the inverse Fourier transform we shall have that (1.30) commutes.

Now observe that

$$\frac{1}{i} \frac{\partial}{\partial z} (\tilde{U}f)(z) = -K \int \frac{\partial Q}{\partial z} e^{-iQ(y,z)} f(y) dy \; ,$$

$$\tilde{U}\left(\frac{1}{i} \frac{\partial}{\partial y} f\right)(z) = K \int e^{-iQ(y,z)} \frac{1}{i} \frac{\partial}{\partial y} f(y) dy$$

$$= K \int \frac{\partial Q}{\partial y} e^{-iQ(y,z)} f(y) dy \quad \text{(by integrating by parts)} \; .$$

Hence

$$\left(-\frac{\partial r}{\partial x'} \frac{1}{i} \frac{\partial}{\partial z} + \frac{\partial r}{\partial \xi'} z\right)(\tilde{U}f)(z) = K \int \left(\frac{\partial r}{\partial x'} \frac{\partial Q}{\partial z} + \frac{\partial r}{\partial \xi'} z\right) e^{-iQ(y,z)} f(z) dz \; ,$$

$$\tilde{U}\left[\left(\frac{\partial r}{\partial \xi} \frac{1}{i} \frac{\partial}{\partial y} + \frac{\partial r}{\partial x} y\right) f\right](z) = K \int \left(\frac{\partial r}{\partial \xi} \frac{\partial Q}{\partial y} + \frac{\partial r}{\partial x} y\right) e^{-iQ(y,z)} f(z) dz \; .$$

Therefore to show that (1.34) commutes it suffices to show

$$\frac{\partial r}{\partial x'} \frac{\partial Q}{\partial z} + \frac{\partial r}{\partial \xi'} z = \frac{\partial r}{\partial \xi} \frac{\partial Q}{\partial y} + \frac{\partial r}{\partial x} y \; . \tag{1.35}$$

Since

$$\frac{\partial r}{\partial x'} = \frac{\partial r}{\partial x} \frac{\partial x}{\partial x'} + \frac{\partial r}{\partial \xi} \frac{\partial \xi}{\partial x'} \; , \qquad \frac{\partial r}{\partial \xi'} = \frac{\partial r}{\partial x} \frac{\partial x}{\partial \xi'} + \frac{\partial r}{\partial \xi} \frac{\partial \xi}{\partial \xi'} \; ,$$

to prove (1.35) it suffices to prove

$$(1.36)\qquad \frac{\partial x}{\partial x'}\frac{\partial Q}{\partial z}+\frac{\partial x}{\partial \xi'}z=y\,,\qquad \frac{\partial \xi}{\partial x'}\frac{\partial Q}{\partial z}+\frac{\partial \xi}{\partial \xi'}z=\frac{\partial Q}{\partial y}\,.$$

Note that since $(x,\xi)\to(x',\xi')$ is *linear*, the coefficients $\partial x/\partial x', \partial x/\partial \xi', \partial \xi/\partial x'$, $\partial \xi/\partial \xi'$ occuring in (1.36) are *constants*, independent of y, z.

Now call $y = x$ and $z = \xi'$, which we may do since x and ξ' are independent variables in the expression $Q(x, \xi')$. Thus we may rewrite (1.36) as

$$(1.36)'\qquad \frac{\partial x}{\partial x'}\frac{\partial Q}{\partial \xi'}+\frac{\partial x}{\partial \xi'}\xi'=x\,,\qquad \frac{\partial \xi}{\partial x'}\frac{\partial Q}{\partial \xi'}+\frac{\partial \xi}{\partial \xi'}\xi'=\frac{\partial Q}{\partial x}\,.$$

Now using (1.32a) we see that (1.36)′ can be rewritten as

$$(1.36)''\qquad \frac{\partial x}{\partial x'}x'+\frac{\partial x}{\partial \xi'}\xi'=x\,,\qquad \frac{\partial \xi}{\partial x'}x'+\frac{\partial \xi}{\partial \xi'}\xi'=\xi\,.$$

But (1.36)″ obviously holds, since $(x,\xi)\to(x',\xi')$ is linear. Thus we have shown that (1.34) commutes. All that remains to be shown is that $\tilde{U}$ is unitary. Since $(x,\xi)\to(x',\xi')$ is linear we may assume that $Q(x,\xi')$ is quadratic in x and ξ'. Thus we may write

$$(1.37)\qquad Q(x,\xi')=\langle Ax,x\rangle+\langle Bx,\xi'\rangle+\langle C\xi',\xi'\rangle\,.$$

Here A, B, C are $n\times n$ matrices, and $\langle\ ,\ \rangle$ denotes the inner product in $\boldsymbol{R}^n$. From (1.32 b) it follows that B is nonsingular. Thus we have

$$\begin{aligned}\|\tilde{U}f\|_2 &= K\left\|\int e^{-i\langle By,z\rangle}e^{-i\langle Ay,y\rangle}f(y)dy\right\|_2\\ &= |\det B|^{-1}K\left\|\int e^{-i\langle y,z\rangle}e^{-i\langle AB^{-1}y,B^{-1}y\rangle}f(B^{-1}y)dy\right\|_2\\ &\qquad\qquad\text{(by changing variables)}\\ &= |\det B|^{-1}(2\pi)^{n/2}K\|f(B^{-1}y)\|_2\quad\text{(by Plancherel)}\\ &= |\det B|^{-1}(2\pi)^{n/2}K|\det B|^{1/2}\|f\|_2\,.\end{aligned}$$

Thus taking $(2\pi)^{n/2}K=|\det B|^{1/2}$ we get that $\tilde{U}$ is unitary. q.e.d.

We now proceed to define the notion of uniform $\frac{1}{2}$-subellipticity for the Poisson complex (1.1) at the point $x_0\in M$. If (1.1) is the top-symbol complex of a complex of first-order pseudo-differential operators, then (1.1) will be $\frac{1}{2}$-subelliptic if and only if the complex of pseudo-differential operators is $\frac{1}{2}$-subelliptic. (For a proof see [4], [9].) First assume that the bundles E^i have Hermitian metrics. If λ converges to x, identify the vector space $H_\lambda(E^i)$ with the subspace $K_\lambda(E^i)\cap(R_\lambda(E^i))^\perp$ of E^i_x, and give it the induced inner product.

Definition 1.11. The Poisson complex (1.1) is $\frac{1}{2}$-subelliptic at x_0 if and only if there is a compact neighborhood $\mathcal{N}$ of x_0 and a constant $C > 0$ such that for every asymptotic sequence $\lambda = \{(x_i, \lambda_i)\}$ with $\lim x_i \in \mathcal{N}$ the following estimate holds:

$$\int |u(y)|^2\, dy \leq C\left\{\int |\mathscr{P}_\lambda u(y)|^2\, dy + \int |\mathscr{P}_\lambda^* u(y)|^2\, dy\right\} \tag{1.38}$$

for every $u \in C_0^\infty(\boldsymbol{R}^n, H_\lambda)$.

It can be shown that the estimate (1.38) is equivalent to the following estimates (1.39) and (1.40) (see [4] or [9]; the essential idea is contained in [6]):

$$\int |u(y)|^2\, dy \leq C\left\{\int |\mathscr{P}_\lambda u(y)|^2\, dy + \int |\mathscr{P}_\lambda^* u(y)|^2\, dy\right\} \tag{1.39}$$

for every $u \in \mathscr{S}(\boldsymbol{R}^n, H_\lambda)$, where $\mathscr{S}$ denotes the Schwartz space;

$$\int |u(y)|^2\, dy \leq C\left\{\int |\mathscr{P}_\lambda u(y)|^2\, dy + \int |\mathscr{P}_\lambda^* u(y)|^2\, dy\right\} \tag{1.40}$$

for every $u \in$ Domain $(\mathscr{P}_\lambda) \cap$ Domain $(\mathscr{P}_\lambda^*)$. Here $\mathscr{P}_\lambda$ is the minimal closed extension of the $\mathscr{P}_\lambda$ appearing in (1.38), and $\mathscr{P}_\lambda^*$ denotes its Hilbert space adjoint. From (1.39) and Lemma 1.10 we see that $\frac{1}{2}$-subellipticity does not depend on the choice of local canonical coordinates. We show below that $\frac{1}{2}$-subellipticity is also independent of the choice of Hermitian metrics for E^i. This is not a priori clear, since $\mathscr{P}_\lambda^*$ is *defined* in terms of these metrics. Assuming this result for the moment, we have

Proposition 2. *If $\{p, E^i\}$ is a Poisson complex on a symplectic manifold M and $x_0 \in M$, then one can define, in a completely intrinsic fashion, the notion of $\frac{1}{2}$-subellipticity at x_0 of $\{p, E^i\}$.*

We are now ready to prove independence of the choice of Hermitian metrics. As we saw earlier in Lemma 1.9 the fact that (1.1) is Poisson implies that the sequence (1.29) is actually a complex. Hence (see [6]) one can show the following:

Lemma 1.12. *The criterion* (1.40) *for $\frac{1}{2}$-subellipticity for the Poisson complex* (1.1) *is equivalent to the following criterion*:

There exists $C > 0$ such that for every asymptotic sequence $\lambda = \{(x_i, \lambda_i)\}$ with $\lim x_i \in \mathcal{N}$, *the L^2-complex*

$$L^2(\boldsymbol{R}^n, H_\lambda(E^0)) \xrightarrow{\mathscr{P}_\lambda} L^2(\boldsymbol{R}^n, H_\lambda(E^1)) \xrightarrow{\mathscr{P}_\lambda} \cdots \xrightarrow{\mathscr{P}_\lambda} L^2(\boldsymbol{R}^n, H_\lambda(E^N)) \tag{1.41}$$

is L^2-exact and the maps $\mathscr{P}_\lambda$ are C-maps. The operators appearing in the complex (1.41) *are the minimal closed extensions of the operators $\mathscr{P}_\lambda$ in the complex* (1.29). *To say that $\mathscr{P}_\lambda$ is a C-map means that for every g in Range $\mathscr{P}_\lambda$ there exists f in Domain $\mathscr{P}_\lambda$ such that $g = \mathscr{P}_\lambda f$ and such that $\|f\| \leq C\|g\|$.*

From this lemma we get

Corollary 1.13. *The $\frac{1}{2}$-subellipticity of the Poisson complex* (1.1) *does not depend on the choice of Hermitian metrics for the bundles E^i.*

Proof. The maps $\mathscr{P}_\lambda$ in the complex (1.29) were defined independently of any choice of Hermitian metrics. Hence, if we make two different choices of Hermitian metrics, all we change is the inner products on the finite dimensional vector spaces E^i_x, and any two such norms are equivalent. One verifies easliy that the equivalence may be made uniform in $\mathscr{N}$. Thus changing Hermitian metrics on the E^i involves at most a change in the choice of uniform constant C in the statement of the lemma. q.e.d.

Aside from the fact that it allows us to intrinsically define the notion of $\frac{1}{2}$-subellipticity of a Poisson complex, the independence of choice of Hermitian metrics allows us to prove the following important result:

Proposition 3. *Suppose the Poisson complex $\{p, E^i\}$ is the direct sum of two subcomplexes $\{p, E^i_a\}$ and $\{p, E^i_b\}$. Then $\{p, E^i\}$ is $\frac{1}{2}$-subelliptic at $x_0 \in M$ if and only if each subcomplex $\{p, E^i_a\}$ and $\{p, E^i_b\}$ is $\frac{1}{2}$-subelliptic at x_0.*

In order for $\frac{1}{2}$-subellipticity of $\{p, E^i_a\}$ and $\{p, E^i_b\}$ to be defined we need the following lemma.

Lemma 1.14. *If the Poisson complex $\{p, E^i\}$ is the direct sum of two subcomplexes $\{p, E^i_a\}$ and $\{p, E^i_b\}$, then each of these subcomplexes is a Poisson complex.*

Proof. Let π^i_a and π^i_b be the complementary projections of E^i onto E^i_a and E^i_b. By hypothesis we have

$$\pi_a p = p\pi_a , \qquad \pi_b p = p\pi_b . \tag{1.42}$$

Choose frames for E^i_a and for E^i_b and use these as frames for E^i. With respect to these frames the matrices π_a and π_b have constant coefficients. Hence we may differentiate (1.42) and get

$$\pi_a \frac{\partial p}{\partial \xi_j} = \frac{\partial p}{\partial \xi_j} \pi_a , \qquad \pi_a \frac{\partial p}{\partial x_j} = \frac{\partial p}{\partial x_j} \pi_a , \tag{1.43}$$

and the analogous identities for π_b. Thus applying π_a on both the left and right of both sides of the identity (1.3), we get

$$\sum \frac{\partial p}{\partial \xi_j} \frac{\partial p}{\partial x_j} \pi_a = (\pi_a q \pi_a) p + p(\pi_a q \pi_a) . \tag{1.44}$$

We also get the analogous identity for π_b. Adding, we get

$$\sum \frac{\partial p}{\partial \xi_j} \frac{\partial p}{\partial x_j} = (\pi_a q \pi_a + \pi_b q \pi_b) p + p(\pi_a q \pi_a + \pi_b q \pi_b) . \tag{1.45}$$

We choose our frames in such a way that

$$(1.46)\qquad \frac{\partial(p\,|\,E_a)}{\partial\xi_j} = \frac{\partial p}{\partial\xi_j}\bigg| E_a\ , \qquad \frac{\partial(p\,|\,E_a)}{\partial x_j} = \frac{\partial p}{\partial x_j}\bigg| E_a\ ,$$

and, similarly, for E_b. Hence by restricting (1.45) to E_a and to E_b we can conclude that $\{p, E^i_a\}$ and $\{p, E^i_b\}$ are Poisson.

Proof of Proposition 3. Since the criterion for $\frac{1}{2}$-subellipticity is independent of the choice of Hermitian metrics we may, having chosen arbitrary metrics for the bundles E^i_a and E^i_b, give the bundle $E^i = E^i_a \oplus E^i_b$ the corresponding direct sum metric. Thus we have the *orthogonal* decomposition

$$(1.47)\qquad H_\lambda(E^i) = H_\lambda(E^i_a) \oplus H_\lambda(E^i_b)\ ,$$

and hence the criterion of Lemma 1.12 for $\frac{1}{2}$-subellipticity clearly holds for $\{p, E^i\}$ if and only if it holds for $\{p, E^i_a\}$ and for $\{p, E^i_b\}$.

2. The top-symbol decomposition

We assume now that the complex (0.1) is a Spencer complex, i.e., formally exact in the sense of Goldschmidt [3]. We consider the top-symbol complex (0.2) as lying over $T^*_{\mathbf{C}}X/\{0\}$ the complexified cotangent space of X.

The main goal of this section is to prove

Proposition 4. *If $(x_0, \zeta_0) \in T^*_{\mathbf{C}}X/\{0\}$ is Cohen-Macaulay, then in some conic neighborhood Γ of (x_0, ζ_0) we can decompose the top-symbol complex* (0.2): $\{\sigma_{(x,\zeta)}(D), E^i_{(x,\zeta)}\}$ *into the direct sum of two subcomplexes*

$$(2.1)\qquad \{\sigma_{(x,\zeta)}(D), (E^i_a)_{(x,\zeta)}\}\ ,$$

$$(2.2)\qquad \{\sigma_{(x,\zeta)}(D), (E^i_b)_{(x,\zeta)}\}\ .$$

Moreover, the complex (2.1) *is exact for every $(x, \zeta) \in \Gamma$, and the complex* (2.2) *is a top-symbol Poincaré complex built out of q commuting bundle maps $p_i\colon E^0_b \to E^0_b$, $i = 1, \cdots, q$, where q is the codimension of the characteristic variety of $\{D, E^i\}$.*

Remarks. 1. Observe that the bundles E^i_a and E^i_b are bundles over the conic neighborhood $\Gamma \subseteq T^*_{\mathbf{C}}X/\{0\}$, and are not necessarily pullbacks of bundles sitting over the manifold X. The splitting is very definitely taking place "upstairs".

2. Clearly, by restriction, the proposition also holds if we replace $T^*_{\mathbf{C}}X/\{0\}$ by $T^*X/\{0\}$. This is the version which we shall use in conjunction with § 1.

We next define the terms "top-symbol Poincaré complex" and "Cohen-Macaulay".

Definition 2.1. Let E^0 be a vector bundle over a manifold M, and suppose that $p_1, \cdots, p_q$ are commuting bundle maps from E^0 to E^0. Then the associated *top-symbol Poincaré complex* $\{p, E^i\}$ is defined as follows: Let W be a q-

dimensional vector space over C, with distinguished basis $w_1, \cdots, w_q$. Then for $i = 0, \cdots, q$ let

$$E^i = E^0 \otimes \bigwedge^i W \,, \tag{2.3}$$

and define $p\colon E^i \to E^{i+1}$ by

$$p\Big(\sum_I e_I \otimes w_I\Big) = \sum_{j,I} p_j e_I \otimes (w_j \wedge w_I) \,, \tag{2.4}$$

where $e_I \in E^0$ and $w_I \in \bigwedge^i W$.

Remark. From the fact that $p_i p_j = p_j p_i$ and that $w_i \wedge w_j = -w_j \wedge w_i$ it follows that $\{p, E^i\}$ is indeed a complex.

Definition 2.2. The cotangent vector (x, ζ) is *characteristic* if and only if the top-symbol complex (0.2) fails to be exact at (x, ζ).

Definition 2.3. The characteristic vector (x, ζ) is *Cohen-Macaulay* if and only if the top-symbol complex (0.2) is exact at E^i for every $i > q$, where q is the codimension of the characteristic variety $\mathscr{V}_x$.

Remarks. 1. The Cohen-Macaulay property is generic, i.e., for every $x \in X$ the set of Cohen-Macaulay points in $\mathscr{V}_x$ is a Zariski-open set.

2. Some hypothesis such as Cohen-Macaulay is clearly needed in our proposition in order to obtain an *exact* subcomplex complementary to a top-symbol Poincaré complex. For example, the Cohen-Macaulay hypothesis prevents the top-symbol complex (0.2) from containing a direct sum of two top-symbol Poincaré complexes of different lengths. Moreover, it is not clear how, without some special hypothesis, one can obtain *any* complementary subcomplex to a top-symbol Poincaré complex. Indeed, in our proof that there exists a complementary subcomplex (2.1) the Cohen-Macaulay hypothesis plays a critical role. It is precisely this hypothesis which allows us to recursively define F^i in Lemma 2.15.

Now let $E = \oplus E^i$ be the direct sum of the bundles in (0.1). Then we may view D as a differential operator from E to E, and we have $D^2 = 0$. Since D is first-order we have that $\sigma(D)(x, \zeta_1 + \zeta_2) = \sigma(D)(x, \zeta_1) + \sigma(D)(x, \zeta_2)$, and since $D^2 = 0$ we have that $(\sigma(D)(x, \zeta))^2 = 0$ for every ζ. Hence we can conclude

$$\sigma(D)(x, \zeta_1)\cdot\sigma(D)(x, \zeta_2) + \sigma(D)(x, \zeta_2)\cdot\sigma(D)(x, \zeta_1) = 0. \tag{2.5}$$

Thus we have the following lemma.

Lemma 2.4. *We can make E_x a (left) graded module over $\bigwedge^*(T_C^*X)_x$, the exterior algebra over $T_C^*(X)_x$, by means of the definitions:*

$$\zeta \wedge e = \sigma_{(x,\zeta)}(D)e \quad \textit{for} \quad e \in E_x \quad \textit{and} \quad \zeta \in T_C^*(X)_x \,, \tag{2.6}$$

$$\zeta \wedge e = \zeta_1 \wedge \cdots \wedge (\zeta_k \wedge e) \quad \textit{for} \quad e \in E_x \quad \textit{and} \quad \zeta = \zeta_1 \wedge \cdots \wedge \zeta_k . \tag{2.7}$$

Starting with the well-known formula

$$D(f\psi) = \sigma_{df}(D)\psi + fD\psi \tag{2.8}$$

we can prove easily

Lemma 2.5. *For every smooth section ψ of E and for every i-form w we have*

$$D(w \wedge \psi) = dw \wedge \psi + (-1)^i w \wedge D\psi . \tag{2.9}$$

Now suppose that we are given a non-characteristic fibration $\pi: X \to Y$ (Y a manifold); i.e., for each point $x \in X$ the complexified normal space N_x to the fiber through x is non-characteristic for $\{D, E^i\}$. That is, each $\zeta \neq 0 \in N_x$ is non-characteristic.

This fibration gives a decrasing filtration of E, namely $\mathscr{T}_j(E^i) \supset \mathscr{T}_{j+1}(E^i)$, where

$$\mathscr{T}_j(E^i)_x = E^i_x \cap (\wedge^j(N_x)E_x) . \tag{2.10}$$

One can show (using Lemma 2.9) that, assuming $\dim_C N_x$ is independent of x, $\mathscr{T}_j(E^i)$ is a vector bundle.

Definition 2.6. We set $\tilde{E}^{0,i}$ = the quotient bundle $E^i/\mathscr{T}_1(E^i)$.

Remark. Observe that $\tilde{E}^{0,0} = E^0$.

From (2.9) we can conclude

Lemma 2.7. *From the complex $\{D, E^i\}$ we obtain an induced complex $\{\tilde{D}, \tilde{E}^{0,i}\}$ of first-order differential operators. Moreover, $\sigma_{(x,\zeta)}(\tilde{D}): \tilde{E}^{0,i}_x \to \tilde{E}^{0,i+1}_x$ is given by*

$$\sigma_{(x,\zeta)}(\tilde{D}): [e] \to [\zeta \wedge e] , \tag{2.11}$$

where $e \in E^i_x$, and [ ] denotes coset in the quotient space.

Remark. From the last statement we see that $\sigma_{(x,\zeta)}(\tilde{D}) = 0$ if $\zeta \in N_x$. That is, $\tilde{D}$ differentiates only in directions tangential to the fibers of π. Hence we may restrict $\{\tilde{D}, E^{0,i}\}$ to any fiber of π, and thus get a complex of differential operators on the fiber. It can be shown that these fiber complexes are also Spencer complexes.

Guillemin [4] proves that if (x_0, ζ_0) is a Cohen-Macaulay point, then it is possible to construct a special non-characteristic fibration. More precisely,

Lemma 2.8. *Let $\{D, E^i\}$ be a Spencer complex on a manifold X of dimension n, with q equal to the codimension of the characteristic variety. Let (x_0, ζ_0) be a Cohen-Macaulay point of the characteristic variety. Then in a neighborhood of x_0 (which we continue to call X)we can define a non-characteristic fibration $\pi: X \to Y$ satisfying the following two properties:*

(1) *The fibers Z of the fibration have codimension q.*

(2) *Let* $\eta_0 = \iota^*(\zeta_0)$, *where* $\iota: Z \to X$ *is the injection into* X *of the fiber of* π *through* x_0. *Then the symbol complex*

$$0 \to \tilde{E}^{0,0}_{x_0} \xrightarrow{\sigma(x_0,\eta_0)(\tilde{D})} \tilde{E}^{0,1}_{x_0} \xrightarrow{\sigma(x_0,\eta_0)(\tilde{D})} \cdots \xrightarrow{\sigma(x_0,\eta_0)(\tilde{D})} \tilde{E}^{0,N}_{x_0} \to 0 \tag{2.12}$$

is exact at every position except at $\tilde{E}^{0,0}_{x_0}$.

Remarks. 1. We can choose a coordinate neighborhood V of x_0 in X, with coordinates $y_1, \cdots, y_q, x_1, \cdots, x_{n-q}$ so that the fibration π takes the form

$$\pi: (y_1, \cdots, y_q, x_1, \cdots, x_{n-q}) \mapsto (y_1, \cdots, y_q) . \tag{2.13}$$

For $x \in V$ and $\zeta \in T^*_C(X)_x$ we shall often write

$$\zeta = (\xi, \eta) = \sum_{k=1}^{q} \xi_k dy_k + \sum_{l=1}^{n-q} \eta_l dx_l . \tag{2.14}$$

Clearly, $\eta = \sum_{l=1}^{n-q} \eta_l \, dx_l$ is $\iota^*(\zeta)$, and N_x is spanned by $dy_1, \cdots, dy_q$. We shall let $\wedge^*_x(dy)$ denote the exterior algebra at x generated by the cotangent vectors $dy_1, \cdots, dy_q$.

2. Since (2.12) holds at (x_0, η_0), it also holds in some conic neighborhood $\tilde{\Gamma}$ of (x_0, η_0) in the bundle of cotangent vectors tangential to the fibers of π. Let Γ equal the set of all cotangent vectors ζ such that $\iota^*(\zeta) \in \tilde{\Gamma}$. Clearly Γ is a conic neighborhood of (x_0, ζ_0).

Guillemin also proves

Lemma 2.9. (1) E_x *is a free graded* $\wedge^*_x(dy)$ *module.*

(2) *Let* $E^{0,i}_x$ *be any subspace of* E^i_x *complementary to* $\mathscr{T}_1(E^i)_x$, *i.e.*, $E^{0,i}_x \oplus \mathscr{T}_1(E^i)_x = E^i_x$. *Then for any* $e_1 \neq 0, \cdots, e_N \neq 0$ *with* $e_i \in E^{0,i}_x$ *and any* $\mu_1, \cdots, \mu_N \in \wedge^*_x(dy)$, *we have* $\mu_1 = \cdots = \mu_N = 0$ *if* $\mu_1 e_1 + \cdots + \mu_N e_N = 0$. *Furthermore,* E_x *and* $\wedge^*_x(dy) \otimes_C (\sum_i E^{0,i}_x)$ *are canonically isomorphic as graded* $\wedge^*_x(dy)$*-modules (in the obvious way).*

Remark. Under the above isomorphism

$$E^k_x \cong \sum_{i+j=k} \wedge^j_x(dy) \otimes E^{0,i}_x , \tag{2.15}$$

$$\mathscr{T}_1(E^i)_x \cong \sum_{\substack{i+j=k \\ j>0}} \wedge^j_x(dy) \otimes E^{0,i} . \tag{2.16}$$

We are now ready to get on with the actual proof of Proposition 4.

Our procedure will be first to construct the top-symbol Poincaré complex (2.2), and second to find a complementary complex (2.1).

Definition 2.10. Let $(x, \eta) \in \tilde{\Gamma}$. Then define

$$\begin{aligned} (E^0_b)_{(x,\eta)} &= \{e \in E^0_x \mid \eta \wedge e \in \mathscr{T}_1(E^1)_x\} \\ &= \{e \in E^0_x \mid \eta \wedge e \in \wedge^1_x(dy) \otimes E^0_x\} . \end{aligned} \tag{2.17}$$

Remarks. 1. $(E_b^0)_{(x,\eta)}$ may be thought of as the "flip-flop" space associated to the *fiber* cotangent vector η. Indeed, it is the set of all e in E_x^0 such that the fiber vector η applied to e is equal to a sum of *base* cotangent vectors applied to elements of E_x^0, i.e.,

$$\eta \wedge e = \sum_{k=1}^{q} dy_k \wedge e_k = \sum_{k=1}^{q} dy_k \otimes e_k \tag{2.18}$$

with $e_k \in E_x^0$, and the second equality being the isomorphism of Lemma 2.9 (2).

2. Clearly $(E_b^0)_{(x,\eta)}$ may vary with η as well as x. This is one important reason why the splitting in Proposition 4 must take place over T_C^*X rather than X.

One sees easily

Lemma 2.11. $(E_b^0)_{(x,\eta)} = \{e \in E_x^0 | \sigma_{(x,\eta)}(\tilde{D})e = 0\}$.

Remark. Using this lemma and the fact that (2.12) holds for every $(x,\eta) \in \tilde{\Gamma}$, we see that E_b^0 has constant rank and hence is a bundle.

Lemma 2.12. $\eta \wedge (E_b^0)_{(x,\eta)} \subseteq \wedge_x^1(dy) \otimes (E_b^0)_{(x,\eta)}$.

Proof. Let $e \in (E_b^0)_{(x,\eta)}$. Referring to (2.18) we want to show that $\eta \wedge e_k \in \wedge_x^1(dy) \otimes E_x^0$ for each k. Applying "$\eta \wedge$" to both sides of (2.18) we get

$$0 = -\sum_{k=1}^{q} dy_k \otimes (\eta \wedge e_k) . \tag{2.19}$$

Choose any subspace $E^{0,1}$ complementary to $\mathscr{T}_1(E^1)_x$. Then we may write, in accordance with (2.15),

$$E_x^1 = E_x^{0,1} \oplus (\wedge_x^1(dy) \otimes E_x^0).$$

For each k, $\eta \wedge e_k \in E_x^1$. We want to show that when we write $\eta \wedge e_k$ in accordance with the above formula the component in $E_x^{0,1}$ equals 0. But this is easy to see from (2.19) and Lemma 2.9 (2).

Definition 2.13. Let $(x,\eta) \in \tilde{\Gamma}$. We define, for $i = 0, \cdots, N$,

$$(E_b^i)_{(x,\eta)} = (E_b^0)_{(x,\eta)} \otimes \wedge_x^i(dy) . \tag{2.20}$$

Lemma 2.14. *Let $(x,\zeta) \in \Gamma$. Then $\zeta \wedge$ maps $(E_b^i)_{(x,\eta)}$ into $(E_b^{i+1})_{(x,\eta)}$, and the complex $\{\zeta \wedge, E_b^i\}$ is a top-symbol Poincaré complex.*

Proof. We write $\zeta = (\xi,\eta)$ where $\xi = \sum_{k=1}^q \xi_k dy_k$ and $(x,\eta) \in \tilde{\Gamma}$. By Lemma 2.12, $\eta \wedge$ maps E_b^0 into E_b^1 and, indeed, we may define bundle maps

$$-a_k(x,\eta) \colon E_b^0 \to E_b^0 \tag{2.21}$$

by $-a_k(x,\eta)e = e_k$, where $\eta \wedge e = \sum_{k=1}^q e_k \otimes dy_k$. We also have

$$\xi \wedge \colon e \mapsto \sum_{k=1}^{q} \xi_k e \otimes dy_k . \tag{2.22}$$

Thus

$$(2.23)\qquad \begin{gathered} E_b^0 \xrightarrow{\zeta\wedge} E_b^0 \otimes \bigwedge{}^1(dy)\ , \\ e \mapsto \sum_{k=1}^{q} (\xi_k - a_k(x,\eta))e \otimes dy_k\ . \end{gathered}$$

Since $\zeta\wedge$ is a $\bigwedge_x^*$-anti-morphism, it is clear from the preceding that

$$(2.24)\qquad \begin{gathered} E_b^i \xrightarrow{\zeta\wedge} E_b^{i+1}\ , \\ e\otimes w \mapsto \sum_{k=1}^{q} (\xi_k - a_k(x,\eta))e \otimes (dy_k \wedge w)\ . \end{gathered}$$

Since $(\zeta\wedge)\cdot(\zeta\wedge) = 0$, it follows that

$$(2.25)\qquad \begin{aligned} &(\xi_k - a_k(x,\eta))\cdot(\xi_l - a_l(x,\eta)) \\ &\quad = (\xi_l - a_l(x,\eta))\cdot(\xi_k - a_k(x,\eta))\ , \end{aligned}$$

and hence $\{\zeta\wedge, E_b^i\}$ is the top-symbol Poincaré complex associated with the q commuting maps

$$\xi_k - a_k(x,\eta)\colon E_b^0 \to E_b^0\ .$$

Remark. Observe that our top-symbol Poincaré complex is completely intrinsic, modulo the choice of non-characteristic fibration π. The complementary complex (2.1) which we shall construct below is not so intrinsic in character.

Before turning to the actual construction of (2.1) we shall first sketch the idea behind it.

By Lemma 2.9 (2), we have that for any choice of $E^{0,i}$ complementary to $\mathscr{T}_1(E^i)$, $i = 1,\cdots,N$, there is a canonical isomorphism between E and $(\sum_i E^{0,i})\otimes\bigwedge^*(dy)$. We also know that, for any choice of $E^{0,i}$, $\xi\wedge$ maps $E^{0,i}$ into $E^{0,i}\otimes\bigwedge^*(dy)$. Suppose it is possible to choose the $E^{0,i}$ in such a way that $\eta\wedge$ maps $E^{0,i}$ into $E^{0,i+1}$, and a subspace E_a^0 complementary to E_b^0 such that $\eta\wedge$ maps E_a^0 into $E^{0,1}$. Then setting

$$(2.26)\qquad E_a^k = (E_a^0\otimes\bigwedge{}^k(dy)) \oplus \Big(\bigoplus_{\substack{i+j=k\\ i>0}} (E^{0,i}\otimes\bigwedge{}^j(dy))\Big)$$

we clearly have a subcomplex (2.1) of the top-symbol complex $\{\zeta\wedge, E^k\}$ complementary to (2.2), i.e.,

$$(2.27)\qquad \zeta\wedge\colon E_a^k \to E_a^{k+1}\ ,\qquad E_a^k\oplus E_b^k = E^k\ .$$

We shall show below that the Cohen-Macaulay hypothesis, in the form of condition (2.12), is precisely the tool we need to construct such $E_{(x,\eta)}^{0,i}$, *varying* *with* η as well as x. Moreover, it will allow us to prove that (2.1) as defined above in (2.26) is exact.

Lemma 2.15. *We can recursively define subspaces* $F^i_{(x,\eta)} \subseteq E^i_x$, $i = 0, \cdots, N$, *such that*

$$E^0_x = (E^0_b)_{(x,\eta)} \oplus F^0_{(x,\eta)} , \tag{2.28}$$

$$E^i_x = \mathscr{T}_1(E^i)_x \oplus \eta \wedge F^{i-1}_{(x,\eta)} \oplus F^i_{(x,\eta)} , \qquad i = 1, \cdots, N. \tag{2.29}$$

Moreover, we can arrange that the F^i *are smooth bundles.*

Assume the lemma for the moment. Then we see immediately that E^0_a and $E^{0,i}$ defined by

$$E^0_a = F^0 , \tag{2.30}$$

$$E^{0,i} = \eta \wedge F^{i-1} \oplus F^i , \qquad i = 1, \cdots, N , \tag{2.31}$$

satisfy the desired properties: $\eta \wedge E^0_a \subseteq E^{0,1}$ and $\eta \wedge E^{0,i} \subseteq E^{0,i+1}$.

Proof of Lemma 2.15. We define $(\tilde{F}^i)_{(x,\eta)} \subseteq \tilde{E}^{0,i}_x$ recursively as follows: Choose $\tilde{F}^0 \subseteq \tilde{E}^{0,0} = E^0$ to be any bundle complementary to E^0_b; and having defined $\tilde{F}^i$, choose $\tilde{F}^{i+1}$ to be a complement in $\tilde{E}^{0,i+1}$ to $\sigma_{(x,\eta)}(\tilde{D})(\tilde{F}^i)$. That is,

$$E^0 = E^0_b \oplus \tilde{F}^0 , \tag{2.32}$$

$$\tilde{E}^{0,i} = \sigma_{(x,\eta)}(\tilde{D})(\tilde{F}^{i-1}) \oplus \tilde{F}^i , \qquad i = 1, \cdots, N . \tag{2.33}$$

Now observe that since $E^0_b = \text{Kernel } \sigma_{(x,\eta)}(\tilde{D})$ and *condition* (2.12) *holds* for $(x, \eta) \in \bar{\Gamma}$, it follows that the complex

$$0 \to \tilde{F}^0_x \xrightarrow{\sigma_{(x,\eta)}(\tilde{D})} \tilde{E}^{0,1}_x \xrightarrow{\sigma_{(x,\eta)}(\tilde{D})} \cdots \xrightarrow{\sigma_{(x,\eta)}(\tilde{D})} \tilde{E}^{0,N}_x \to 0 \tag{2.34}$$

is exact. From (2.33) and the exactness of (2.34) we conclude:

Sublemma. $\sigma_{(x,\eta)}(\tilde{D})$ *is injective on* $\tilde{F}^i$ *for every* $i = 0, \cdots, N$.

Next choose $(F^i)_{(x,\eta)} \subseteq E^i_x$ such that

$$[F^i] = \tilde{F}^i , \tag{2.35}$$

(where [ ] denotes coset in $\tilde{E}^{0,i}$) and such that

$$F^i \cap \mathscr{T}_1(E^i) = \{0\} . \tag{2.36}$$

It is easy to see that we can find such F^i, and can even choose them to be smooth bundles. We can take $F^0 = \tilde{F}^0$, and so (2.32) becomes

$$E^0 = E^0_b \oplus F^0 .$$

Moreover, recalling (2.11) we see, from (2.33) and the sublemma and from the way we defined F^i, that

$$E^i = \mathscr{T}_1(E^i) \oplus \eta \wedge F^{i-1} \oplus F^i\,, \qquad i = 1, \cdots, N\,.$$

q.e.d.

To complete the proof of Proposition 4 it suffices to show that the complex (2.1) constructed above is exact for every $(x, \zeta) \in \Gamma$. Define

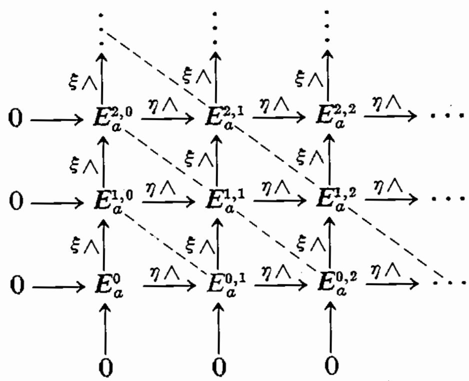
$$E_a^{j,i} = \begin{cases} E^{0,i} \otimes \wedge^j(dy) & i > 0\,, \\ E_a^0 \otimes \wedge^j(dy) & i = 0\,. \end{cases} \tag{2.37}$$

Then

$$E_a^k = \bigoplus_{i+j=k} E_a^{j,i}\,, \tag{2.38}$$

where we think of j as the filtration degree. Writing $\zeta = (\xi, \eta)$ we see that $\xi \wedge$ raises filtration degree by one, and $\eta \wedge$ does not raise filtration degree. We have the diagram:

(2.39)



Lemma 2.16. *The complex* (2.1)

$$0 \to E_a^0 \xrightarrow{\zeta\wedge} E_a^1 \xrightarrow{\zeta\wedge} \cdots \xrightarrow{\zeta\wedge} E_a^N \to 0$$

is exact for every $(x, \zeta) \in \Gamma$.

Proof. *Case* 1: $\xi \neq 0$. Since ξ is non-characteristic, all the columns in diagram (2.39) are exact at every position. Hence by a standard spectral sequence argument all the cohomology groups of the complex (2.1) vanish.

Case 2: $\xi = 0$. By (2.11) the complex $\{\eta \wedge, E_a^{0,i}\}$ is just the complex (2.34) brought "off the quotient level", and we observed earlier that (2.34) is exact. Since each row of diagram (2.39) is just a direct sum of copies of $\{\eta \wedge, E_a^{0,i}\}$, it follows that $\{\eta \wedge, E_a^i\}$ is exact. q.e.d.

This completes the proof of Proposition 4.

3. Conclusion: The Levi-form criterion for $\frac{1}{2}$-subellipticity

One sees easily that an exact Poisson complex is always $\frac{1}{2}$-subelliptic. One can also prove the following proposition (see [4] or [9]; this is also essentially contained in [7]):

Proposition 5. *Let $\{p, E^i\}$ be a top-symbol Poincaré complex associated to scalar bundle maps $p_1, \cdots, p_q$ from E^0 to E^0;* i.e., *p_i is a scalar times the identity matrix. Assume that $\{p, E^i\}$ is Poisson. Then a necessary and sufficient condition for $\{p, E^i\}$ to be $\frac{1}{2}$-subelliptic at position E^k at the real characteristic point (x_0, ξ_0) is the following:*

(3.1) *The $q \times q$ Hermitian matrix $(1/\sqrt{-1})\{p_i, \bar{p}_j\}_{(x_0,\xi_0)}$, the "Levi-form", has at least $k+1$ negative or at least $q-k+1$ positive eigenvalues.*

In [5] Guillemin gives conditions, which we shall loosely call "simple characteristics", under which the top-symbol Poincaré complex derived in Proposition 4 is scalar. Hence from Proposition 3, 4, and 5 we see that under the assumption of simple characteristics we get a Levi-form criterion for $\frac{1}{2}$-subellipticity for a Spencer complex $\{D, E^i\}$. In fact, one can show that, under the simple characteristics hypothesis, we may take as the $p_1, \cdots, p_q$ appearing in the Levi-form any set of local parameters at (x_0, ξ_0) for the complex characteristic variety. (See [5] for a detailed statement.)

References

[1] J. M. Boardman, *Singularities of differentiable maps,* Inst. Hautes Études Sci. Publ. Math. No. 33 (1967) 21–57.

[2] C. Caratheodory, *Calculus of variations and partial differential equations of the first order*, Part I, Holden Day, San Francisco, 1965.

[3] H. Goldschmidt, *Existence theorems for analytic linear partial differential equations,* Ann. of Math. **86** (1967) 246–270.

[4] V. W. Guillemin, unpublished.

[5] V. W. Guillemin & S. Sternberg, *Subelliptic estimates for complexes,* Proc. Nat. Acad. Sci. U.S.A. **67** (1970) 271–274.

[6] L. Hörmander, L^2 *estimates and existence theorems for the* $\bar{\partial}$ *operator,* Acta Math. **113** (1965) 89–152.

[7] ——, *Pseudo-differential operators and nonelliptic boundary problems,* Ann. of Math. **83** (1966) 129–209.

[8] L. Hörmander & J. J. Duistermaat, *Fourier integral operators.* II, Acta Math. **128** (1972) 183–269.

[9] C. Rockland, *A new construction of the Guillemin-Poincaré complex,* Thesis, Princeton University, Princeton, 1972.

[10] W. J. Sweeney, *A characterization of local subellipticity for pseudo-differential complexes,* J. Differential Equations **13** (1973) 495–505.

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